Oscillating shells: A model for a variable cosmic object

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Abstract

A model for a possible variable cosmic object is presented. The model consists of a massive shell surrounding a compact object. The gravitational and self-gravitational forces tend to collapse the shell, but the internal tangential stresses oppose the collapse. The combined action of the two types of forces is studied and several cases are presented. In particular, we investigate the spherically symmetric case in which the shell oscillates radially around a central compact object.

Subject headings: Thin Shell, Oscillation, Dynamics-Stars

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1 Introduction

Since the pioneering works of Ostriker and Gunn (Ostriker and Gunn, 1977), the study of the motion of thin clouds surrounding an exploding star has been the basis for understanding the dynamic process occurring after a supernova explosion, the motion of a blast wave (Sato, 1993) and several other related phenomena. These studies were done in Newtonian theory, where an equation of motion derived from Newton's second law was used, introducing the forces that act on the shell. For the case of a supernova we have as the interacting forces the gravitational force, the force due to the radiation pressure emitted by the star, and a friction type force due to the interaction of the shell with the interstellar medium. The deduction is clear and straight-forward and the picture obtained is quite useful and has a good degree of accuracy for several cases in describing the main processes occurring during the motion.

On the other hand, in General Relativity, the relativistic version of the equation of motion of a shell has its origins in studies of the conditions needed to join two regions of space-time, the so called matching conditions. Initially, the Lichnerowicz matching conditions (Lichnerowicz, 1955) gave conditions on the boundary between two regions of the space time, where the boundary was just the end of one region and the beginning of the next. Israel (Israel, 1966, 1967) gave a different interpretation to this boundary, allowing it to be a thin shell of matter, thus the two regions of the space time are matched on this boundary, the thin shell, obtaining new matching conditions, for which, when the shell parameters tend to zero, the Lichnerowicz ones are recovered. Israel's matching conditions for two spaces can be interpreted as the equations that govern the motion of the shell in order to consistently separate the two given space times, that is, can be read as the equations of motion for the shell, which has had very different applications, such as allowing the study of collisions of shells (Núñez et al. 1993). Recently Núñez and Oliveira (Núñez and Oliveira 1996) explicitly obtained the general relativistic version of the Newtonian equation of motion. Thus, both theories are now on the same footing for analyzing the motion of a shell.

Even though the Relativistic analysis demands for its formulation a more complicated mathematical machinery and often the results are negligible corrections to the Newtonian description, the theory has several conceptual ad-

vantages and applications that make it worth considering as an alternative approach to analyzing the motion of shells. It is definitely necessary to use it in cases where the gravitational fields are strong, with high values of the density compared to the pressure or when the velocities are close to the velocity of light. With respect to the motion of the shell, it is necessary to use the Relativistic formalism when the shell surrounds a black hole or when the shell itself is very dense and moves at high speeds. The model presented in this work belongs to this last case. Moreover, using the fact that in General Relativisty the explicit equations relating the geometrical quantities (the "forces" of the Newtonian theory) with the matter energy distribution in the space are given (the Einstein equations!), the set of equations of motion can be reduced to a set of first order differential equations, that is, a first integral of the equations of motion can be obtained, which in the Newtonian limit is related to the energy of the system. The direct deduction of this first integral for the case of spherical symmetric space times with a matter shell of perfect fluid was done by Núñez and Oliveira, (Núñez and Oliveira 1996).

In the present work we use this set of first order differential equations plus an equation of state for the surface density and the tangential pressure of the shell, in order to formulate a model in which the shell moves due to the combined action of the tangential stress and the gravitational attraction. With an appropriate selection of the parameters, different types of motion can be described. We present four cases of motion, an indefinite expansion of the shell starting from rest, which can be seen as an explosion; two cases where the shell oscillates and a case where the shell is static. The oscillation case is particularly interesting because it implies a change in the shell's surface density which in turn could be related to a change in temperature, which would look like an object of variable magnitude to a distant observer. In all cases, the values of the parameters of the shell obtained, the tangential pressure and the surface energy are large. They are the two dimensional analogy of the values of a neutron star, so this models represent the motion of a sort of "neutron shell". Finally, we have to call the reader's attention to the fact that the equation of state used in the derivation of the oscillatory motions and the expansion one is quite unusual and it is unlikely that it describes a realistic type of matter, so the model should be taken as a first approach. This is not so for the static case, where the result is valid for all equations of state.

Even though in order to analyze the oscillation of an object such as a

Cepheid star one would need to take into account more parameters, we think that the type of analysis presented here is in the correct direction in obtaining an analytic model for such objects.

The paper is organized as follows: In the next section we present the equations of motion. A detailed discussion of the projections of the geometrical quantities form both spaces on the shell and the deduction of the junction conditions is presented in the appendix. We also present a brief analysis of the stability of the shells under radial perturbations. In section 3 we introduce an expression for the tangential pressure on the shell, obtain the equation of state and construct the model; presenting and analyzing several cases for different sets of parameters. We also include the static case which, as we said, is independent of the equation of state. Finally in section 4 we give some conclusions and mention possible directions for further research.

2 Equations of Motion and Stability Analysis

As was mentioned above, in the General Relativistic formalism the motion equations of the shell are obtained as matching conditions between the two space-times that the shell separates. In the case when these space-times are spherical symmetric and the matter - energy of the shell is described by a perfect fluid, the equations of motion reduce to a set of two coupled first order differential equations.

The general form of the line element of a spherical symmetric space time is given by:

$$ds^2 = -e^{\psi} dv((1 - \frac{2\,m}{r})\, e^{\psi} dv - 2dr) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where v is the advanced null coordinate, m and ψ are arbitrary functions of v and r. We are using the usual units in General Relativity, where the speed of light and the gravitational constant are equal to unity, but in the concrete examples we will give the quantities in the international system of units as well. For this symmetry the equations of motion are, the mass-energy conservation equation:

$$\dot{M} + p\,\dot{A} = A\,[T_{\mu\nu} n^{\mu}\, u^{\nu}], \tag{2}$$

where M is the mass of the shell (not necessarily constant!), p is its two - dimensional tangential pressure (it is n̲ot the ordinary 3 - dimensional

pressure) and $A = 4\pi\, R^2(\tau)$ is the area of the shell, related to the shell's density σ by $\sigma = \frac{M}{A}$, (see Appendix). A quantity in squared brackets stands for the difference of that quantity evaluated outside the shell, $+$, minus the quantity evaluated inside, $-$. Dot, " " denotes differentiation with respect to the proper time of the shell, τ, given in Eq. (A.8).

The second equation is the dynamic equation for the radius of the shell:

$$\dot{R}^2 + V(R) = 0, \tag{3}$$

where $V(R)$ is the potential given by

$$V(R) = 1 - (\frac{m_+ - m_-}{M})^2 - \frac{m_+ + m_-}{R} - (\frac{M}{2\,R})^2, \tag{4}$$

The Einstein equation for the line element given above are for each space time

$$m_{,v} = 4\,\pi\,r^2\,T^r{}_v, \qquad m_{,r} = -4\,\pi\,r^2\,T^v{}_v,$$
$$\psi_{,r} = 4\,\pi\,r\,T_{rr}, \tag{5}$$

To completely determine the motion of the shell, one has to supply an equation of state for the matter of the shell which relates the tangential pressure with the superficial density, $p = p(\sigma)$. In this way, the dynamic problem for determining the motion of the shell between two given spaces consist of two first order coupled differential equations.

A typical problem could be posed as follows: First determine where the shell is going to be moving, that is, give the metric tensors $g^+_{\mu\nu}$, and $g^-_{\mu\nu}$ of the two regions of the space-time, $\mathcal{M}_+$ and $\mathcal{M}_-$ respectively, which is equivalent to giving the matter - energy distribution of each region of the space time through the Einstein's equations . Second we choose a equation of state for the matter on the shell, we specify what it is composed of, $p = p(\sigma)$. This information completely specifies all the parameters in the equations of motion, so we can proceed to their analysis.

Notice that the formalism to study the motion of the shell has been developed to the point that no more deduction of equations is needed, we need only define the parameters and proceed directly to study the equations of motion.

Finally, it is of interest to say some words about the stability of the shells. We will restrict ourselves to the stability analysis with respect to radial

perturbations, which are most important because of the spherical symmetry and describe how the shell reacts to the influence of radial deviations.

We start from the dynamic equation for the radius of the shell, Eq. (3), and perform a variation $R(\tau) \to R(\tau) + \delta R(\tau)$, which, to first order in $\delta R(\tau)$ implies:

$$\dot{R}^2(\tau) \to \dot{R}^2(\tau) + 2\,\dot{R}(\tau)\,(\delta R(\tau))^{\cdot}, \tag{6}$$

and for the potential, $V(R)$, we have that $V(R) \to V(R+\delta R)$, and a Taylor expansion around $V(R)$ to first order in δR gives (in what follows we do not show explicitly the dependence on τ):

$$V(R) \to V(R) + \frac{\partial V}{\partial R}|_R\,\delta R. \tag{7}$$

Substituting these last two equations in Eq. (3) and using the fact that we are perturbing a solution of this equation, we get:

$$2\,\dot{R}\,(\delta R)^{\cdot} + \frac{\partial V}{\partial R}|_R\,\delta R = 0, \tag{8}$$

which is the equation for the evolution of the radial perturbation. Again using Eq. (3), expressed as $\dot{R} = \sqrt{-V}$, which on integration yields:

$$\delta R = \exp \int \frac{\partial \sqrt{-V}}{\partial R}|_R\,d\,\tau. \tag{9}$$

Now the integrand, using Eq. (4), has the form:

$$\frac{\partial \sqrt{-V}}{\partial R}|_R = \frac{-\frac{2\,(m_+ + m_-)\,R + M^2}{2\,R^3} + (\frac{M^4 - 2\,(m_+ - m_-)^2\,R^2}{2\,M^3\,R^2})\frac{\partial M}{\partial R}}{2\sqrt{-V}}. \tag{10}$$

In order to proceed further we would need to specify the model chosen and then obtain explicitly R and M as functions of τ, perform the integration and study how the perturbation behaves. Fortunately, some general remarks can be made about the integrand so something can be said even without an explicit solution. We will return to this point in the next section.

3 The Model

In the present work we want to show the action of two combined forces acting on the shell and how their interaction produces several types of motion, including oscillatory motion. We will construct a simple model where there will be vacuum outside the shell, and a compact object in vacuum inside of it, that is, we will take the outside and inside space times to be described by the Schwarzschild metric, with constant gravitational masses m_+ and m_- respectively, and $T^{\pm}_{\mu\nu} = 0$. This will generate a gravitation attraction towards the center, so it accounts for the inward force.

With respect to the outward force, consider that, as it collapses the density of the shell grows, which in turn produces an increase in the tangential pressure, which generates a roman arch type of force that opposes the collapse. In the present model, we will represent the growth of the tangential pressure between particles of the shell with dependence on the radius of the shell by the expression

$$p = p_0 \, e^{-\kappa \, R}, \tag{11}$$

with p_0 and κ constant parameters, and $R = R(\tau)$ is the radius of the shell at the time τ, the proper time of the shell.

With these suppositions the mass-energy conservation equation of the shell, Eq. (2), becomes

$$\dot{M} = -8 \, \pi \, p_0 \, R \, e^{-\kappa \, R} \dot{R}, \tag{12}$$

which can be solved in terms of R:

$$\tilde{M} = \tilde{M}_A + \frac{8 \, \pi \, p_0}{\kappa^2 \, M_\odot} \, (1 + \kappa \, R_\odot \, \tilde{R}) \, e^{-\kappa \, R_\odot \, \tilde{R}}, \tag{13}$$

with $\tilde{M}_A = const.$, the "dust mass" of the shell, and we are expressing the masses as multiples of the Solar mass, $M_\odot$, and the radius R as multiples of the solar radius $R_\odot$, that is $M = M_\odot \, \tilde{M}$, $R = R_\odot \, \tilde{R}$. $\tilde{M}$ and $\tilde{R}$ are unitless. We can go further and obtain an equation of state for the matter of the shell,

$$\sigma = \frac{M_\odot \, \tilde{M}_A \, \kappa^2 + 8 \, \pi \, (1 - \ln(\frac{p}{p_0})) \, p}{4 \, \pi \, (\ln(\frac{p}{p_0}))^2}. \tag{14}$$

This equation of state is rather unusual, but it is well behaved for $p > p_0$, which is always the case in the motion studied, since the shell never reaches

a zero radius. Besides, as will be shown below in the examples, it does not violates the energy conditions, namely $|p| < \sigma$ is always satisfied so the matter is not of an "exotic" type. Still, we have to agree that it is unlikely that the matter defined by such a equation of state would be realistic, so the model presented here should be taken just as a first approach.

In the dynamic equation for the shell, Eq. (3), the potential, Eq. (4), has the gravitational masses m_- inside and m_+ outside, which are constants, and the gravitational mass of the shell is given by Eq. (13). So in terms of the solar parameters we have for the potential the following expression:

$$V = 1 - (\frac{\tilde{m}_+ - \tilde{m}_-}{\tilde{M}})^2 - (\frac{M_\odot}{R_\odot})\,\frac{\tilde{m}_+ + \tilde{m}_-}{\tilde{R}} - (\frac{M_\odot}{R_\odot})^2\,(\frac{\tilde{M}}{2\,\tilde{R}})^2, \tag{15}$$

and with the units we are using, $G = c = 1$, both the Solar mass and the Solar radius are in length units, $M_\odot = 1.473 * 10^5\,\mathrm{cm}$, and $R_\odot = 6.95 * 10^{10}\,\mathrm{cm}$. The constants κ and p_0 have units of cm^{-1}.

The dynamic equation, Eq. (3), is the expression for the total conserved energy of the shell which, as it is characteristic in General Relativity, does not have an arbitrary value but a fixed one, (in our case zero), so the kinetic energy is equal to minus the potential energy for all the motion. From this fact we can conclude several properties of the potential term. First, as the kinetic energy has always to be positive, the potential in a well defined motion has to be negative over all the range of radius. Also, wherever the shell stops, the potential energy term has to be equal to zero at those points and vice versa. A minimum for the potential corresponds to a maximum of the kinetic energy and of course there can be only one minimum between the turning points and the second derivative or the potential with respect to the radius has to be positive. From inspection of the potential equation we can also conclude that far from the shell, the leading behavior of the potential is $1 - (\frac{\tilde{m}_+ - \tilde{m}_-}{\tilde{M}_A})^2$. Finally, from the relation between the Solar mass and the Solar radius we expect a relation between the values of the radius, R, and the values of the masses, m_+, m_-, M of the order of 10^6. Thus the problem is posed as a five-parameter dynamic problem.

With these general considerations, even though the motion equation can not be solved analytically, we can search for parameter sets which determine what motions could be described by our model.

As a matter of fact, there is a large range of values of the parameters for which several types of motion can occur. We present three cases, two of

oscillatory motion, one with masses of the order of unity, and another with radius of the order of unity, and a case with indefinite expansion

Finally, we present a static case, in which the shell is at rest. This case is particularly interesting because the mass energy conservation equation is satisfied directly and we do not need to specify an equation of state and the results for this model hold for all equations of state.

With respect to the stability analysis, we can give also some general remarks. As mentioned above, the motion is defined for a range of radius where the potential is negative, so the integrand for the radial perturbation, Eq. (9), is real for the range where the motion is taking place so the perturbations either increase exponentially or decrease exponentially, but we do not expect oscillatory type of perturbations. For the cases where the shell oscillates or expands, we would need the explicit solution of the motion to say something more definite. Again, this is not the case for the static model where the perturbation analysis has to be taken to the second order of the perturbations, where the perturbation equation can be solved as we show below.

Now we proceed to present the different cases:

i) With values for the masses of the order of solar masses, we take for the Schwarzschild mass outside, $\tilde{m}_+ = 1.1$, that is $m_+ = 2.18 * 10^{33}$g.; for the Schwarzschild mass inside, $\tilde{m}_- = 0.5$, that is $m_- = 9.93 * 10^{32}$g.; for the gravitational "dust mass" of the shell, $\tilde{M}_A = 0.603$, that is $M_A = 1.198 * 10^{33}$g.; for the pressure constant $p_0 = 10^{-3}cm^{-1}$; and for the constant $\kappa = 2.8 * 10^{-6}\text{cm}^{-1}$.

These values generate a potential of the form shown in Figure 1. Notice the region below the R-axis, where the potential takes negative values which, as we explained above, is the region where the motion is allowed. We have two crossing points with the R-axis where the potential equals zero, so the motion is bounded by these two values of the radius, which are $\tilde{R}_{\min} = 9.0847 * 10^{-5}, (R_{\min} = 6.313 * 10^6\text{cm.})$; and $\tilde{R}_{\max} = 3.415 * 10^{-4}, (R_{\max} = 2.373 * 10^7\text{cm.})$. At these extremes the shell stops and starts moving in the oppsosite direction; in Figure 2 we present a graph of the shell's velocity, reminding the reader that these velocities are given as factor of the velocity of light which in these units has a value of 1, ($c = 3 * 10^{10}\frac{\text{cm}}{\text{sec}}$). Notice that the maximum velocity of the shell occurs closer to the minimum radius, where the tangential pressure acts strongly and decelerates the shell, finally stopping it. At that point the velocity reaches a maximum with a value of the order of 0.14. Taking the average velocity to be roughly half this

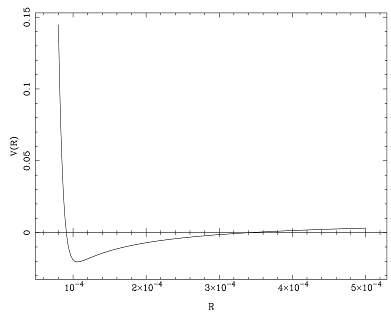

Figure 1: Potential for the motion of the shell in the case of masses of the order of solar masses. $\tilde{R}$ is given a multiples of the solar radius, the potential $V(\tilde{R})$ has no units.

maximum, $\overline{v} = 0.07$, and recalling that the one-way distance covered is $d = R_{\rm max} - R_{\rm min} = 17.4 * 10^6$cm., we obtain for the period of the oscillation, $T = 1.65 * 10^{-2}$s.

Finally, from the expression for the surface density, Eq. (14), and from that for the tangential pressure, Eq. (11), we can obtain the range of values over which these functions vary:

$$\begin{aligned} \sigma &\in [1.798 * 10^{-10}, 1.254 * 10^{-11}]\,\mathrm{cm}^{-1} = \\ &\quad [2.426 * 10^{18}, 1.693 * 10^{17}]\,\frac{\mathrm{g}}{\mathrm{cm}^2}, \qquad (16) \\ p &\in [2.0998 * 10^{-11}, 1.3758 * 10^{-32}]\,\mathrm{cm}^{-1} = \\ &\quad [2.549 * 10^{38}, 1.670 * 10^{17}]\,\frac{\mathrm{dyne}}{\mathrm{cm}}, \qquad (17) \end{aligned}$$

where in the brackets we give the values that the function takes at the minimum radius and at the maximum. We want to stress the fact that in the whole range the strong energy condition, $\sigma > |p|$, holds which means that the energy density is positive definite for all observers so, as we mentioned, even though the equation of state is unusual, it does describe a well defined type of matter. Finally, we remind the reader that these values of density and pressure are defined on a surface, so the comparison of their magnitudes with quantities defined on volumes is not well posed. Nevertheless, we can say that we are talking about large densities and stresses, the "neutron shell" that we mentioned.

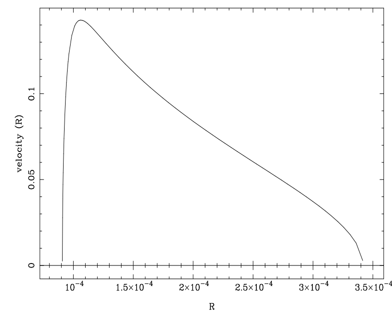

Figure 2: Shell velocity for the potential shown in Figure 1. The velocity is unitless, as multiples of the speed of light, $c = 1$.

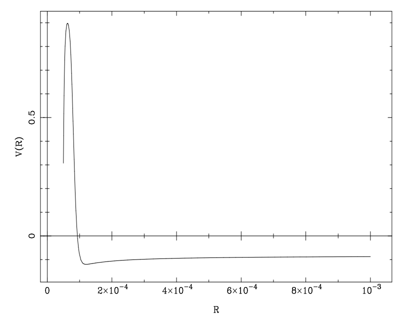

Figure 3: Potential for the case of an indefinite motion, the shell starts from rest at a radius of $\tilde{R}_o = 9.314 * 10^{-5} R_\odot$.

ii) As was discussed above, for large radius the potential tends to a constant value given by $1 - (\frac{\tilde{m}_+ - \tilde{m}_-}{\tilde{M}_A})^2$, so choosing for the outside Schwarzschild mass, $\tilde{m}_+ = 1.5$, ($m_+ = 2.98 * 10^{33}$g.), for the inside Schwarzschild mass, $\tilde{m}_- = 0.73$, ($m_- = 1.45 * 10^{33}$g.); and for the gravitational "dust mass" of the shell, $\tilde{M}_A = 0.74$, ($M_A = 1.47 * 10^{33}$g.), the potential tends to the value $V \to -4.05 * 10^{-2}$; that is, it expands indefinitely with a constant velocity of $v \to 0.201c = 6 * 10^4 \frac{\mathrm{Km}}{\mathrm{s}}$. This behavior represents an explosion. With the values of $p_0 = 5 * 10^{-4}\,\mathrm{cm}^{-1}$, $\kappa = 2.38 * 10^{-6}\,\mathrm{cm}^{-1}$, the potential is shown in Figure 3. The present case can be interpreted as a shell starting from rest at a radius of $R_o = 9.34 * 10^{-5} = 6.473 * 10^6$cm from the center, and the tangential pressure expels it in such a way that the gravitational attraction cannot stop the motion, so the shell continues expanding indefinitely, tending towards a uniform motion with constant velocity.

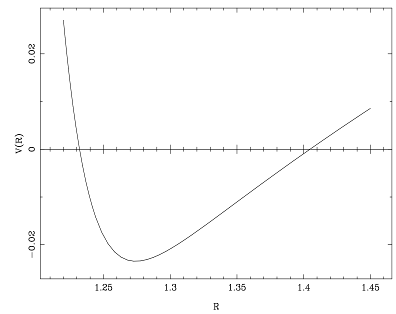

Figure 4: Potential for the motion of the shell for the case of turning point radius of the order of one solar radius.

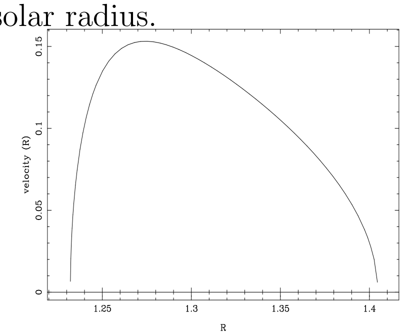

Figure 5: Shell velocity for the potential shown in Figure 4.

iii) Choosing the parameters in such a way that the shell oscillates within values of the radius of the order of a solar radius, we take $\tilde{m}_+ = 1.3 * 10^5, \tilde{m}_- = 4.45 * 10^4, \tilde{M}_A = 10^5 (m_+ = 2.59 * 10^{38}\,\mathrm{g.}, m_- = 8.51 * 10^{37}\,\mathrm{g.} M_A = 1.989 * 10^{38}\,\mathrm{g.})$, and taking $p_0 = 5 * 10^{14}\,\mathrm{cm}^{-1}, \kappa = 7.41 * 10^{-10}\,\mathrm{cm}^{-1}$. In this way we obtain a potential which again describes oscillatory motion. The region of the potential with negative values is presented in Figure 4. The minimum and maximum radius of the oscillation, the turning points, are $R_{\mathrm{min}} = 1.2321\, R_\odot = 8.56 * 10^{10}\,\mathrm{cm.}$, $R_{\mathrm{max}} = 1.4045\, R_\odot = 9.76 * 10^{10}\,\mathrm{cm}$ which, as we wanted, are of the order of a Solar radius.

A graph of the shell's velocity is presented in Figure 5. Notice that the maximum velocity of the shell is $v_{\mathrm{max}} = 0.15c$ (which is of the order of the maximum velocity for the first case presented). Taking the average as $\overline{v} = 0.07c$, we obtain that the oscillation period for this case is $T = 31.5\mathrm{s}$.

Finally, for the surface density and the tangential pressure we obtain that their values change between the minimum radius and the maximum in the

range

$$
\begin{aligned}
\sigma &\in [1.6429 * 10^{-13}, 1.231 * 10^{-13}]\,\mathrm{cm}^{-1} = \\
&\quad [2.21 * 10^{15}, 1.661 * 10^{15}]\,\frac{\mathrm{g}}{\mathrm{cm}^2}, \qquad (18) \\
p &\in [1.386 * 10^{-13}, 1.9823 * 10^{-17}]\,\mathrm{cm}^{-1} = \\
&\quad [1.682 * 10^{36}, 2.406 * 10^{32}]\,\frac{\mathrm{dyne}}{\mathrm{cm}}, \qquad (19)
\end{aligned}
$$

again the strong energy condition holds over the whole range and we obtain large values for the surface density and the tangential stress.

Static case. The equations of motion in our model also allow for a static solution. In this case we demand that the gravitational and stress forces cancel each other. This implies a constant gravitational mass of the shell, M, (see Eq. (2)). Notice that we do not need to specify the equation of state for the matter on the shell. Our only claim is that M is constant so this model holds for all equations of state.

Since $\dot{R} = 0$, then by the equation of motion (3) the potential must also be zero. Equating to zero the expression for the potential, Eq. (15), with the all masses taken as constants, we obtain an expression for the constant radius which satisfies this equation, that is, the static radius, $R_{\rm st}$:

$$\tilde{R}_{\rm st} = \frac{M_\odot\,\tilde{M}^2}{2\,R_\odot}\,(\frac{\tilde{m}_+ + \tilde{m}_- \pm \sqrt{\tilde{M}^2 + 4\tilde{m}_+\tilde{m}_-}}{\tilde{M}^2 - (\tilde{m}_+ - \tilde{m}_-)^2}). \tag{20}$$

Thus, at this radius the shell remains at rest. If we choose a shell of mass $\tilde{M} = 0.1, M = 1.98 * 10^{32}\,$g., surrounding a star with a mass such as that of the Sun, $\tilde{m}_- = 1, m_- = 1.98 * 10^{33}\,$g., and taking for the exterior space a gravitational mass $\tilde{m}_+ = 1.05, m_+ = 2.088 * 10^{33}\,$g., then we obtain the value of $R_{\rm st}$ for which the shell is static surrounding this object,

$$R_{\rm st} = 5.7956 * 10^{-6}\,R_\odot = 40.27\text{km}. \tag{21}$$

For the object in the interior we can choose any radius less that the static one and greater than Schwarzschild horizon of the space times or that of the shell, (these radii are, for the values of the gravitational masses chosen, $r_{\rm Sch_+} = 3.09$km. $> r_{\rm Sch_-} = 2.945$km. $> r_{\rm Sch_M} = 0.294$km.) It could be a neutron star with a typical radius of $R = 15.12$km. (S. Shapiro and P. Teukolsky, 1983).

The stability analysis for this case has to be carried out to the second order perturbations, as the first order one, Eq. (9), evaluated at the static solution implies $\frac{\partial V}{\partial R}|_{R_{st}} = 0$ and does not give us information about the perturbation. It seems more convenient to analyze the perturbations starting from the second order differential equation for the motion:

$$\ddot{R} = -\frac{\partial V}{2\,\partial R}, \tag{22}$$

which implies for the perturbation δR:

$$(\ddot{\delta R}) = -\frac{\partial^2 V}{2\,\partial R^2}|_{R_{st}}\,\delta R, \tag{23}$$

an harmonic oscillation equation. It is important to notice that, even though for the static case $\dot{M} = 0$, this does not implies $\frac{\partial M}{\partial R} = 0$. Actually, for each static radius, the mass of the shell changes accordingly. In obtaining this variation of the mass of the shell with respect to the static radius, we use again the static condition, $V(R) = 0$, taking it as an equation for $\tilde{M} = \tilde{M}(\tilde{R}_{st})$, which results in a fourth order algebraic equation for $\tilde{M}$, with roots:

$$\tilde{M} = \frac{R_\odot}{M_\odot} \sqrt{2\,\tilde{R}_{st}\,[\tilde{R}_{st} - (\tilde{m}_+ + \tilde{m}_-)\,(M_\odot/R_\odot) \pm \sqrt{\Delta}]}, \tag{24}$$

where $\Delta = (\tilde{R}_{st} - 2\,\tilde{m}_+\,(M_\odot/R_\odot))\,(\tilde{R}_{st} - 2\,\tilde{m}_-\,(M_\odot/R_\odot))$, and we are ignoring the clearly negative roots. From this last equation for $\tilde{M}$, we obtain that

$$\frac{\partial M}{\partial R_{st}} = \pm \frac{M\,(\tilde{R}_{st} \pm \sqrt{\Delta})}{2\,R_{st}\,\sqrt{\Delta}}. \tag{25}$$

If we substitute this last expression in Eq.(10), that amounts to evaluate $\frac{\partial V}{\partial R}$ at the static radius, we obtain that $\frac{\partial V}{\partial R}|_{R_{st}} = 0$, as expected. Now we proceed to analyze the second derivative of the potential $V(R)$, which is given by:

$$\begin{aligned}\frac{\partial^2 V}{\partial R^2} = \; & -\frac{2\,(\tilde{m}_+ + \tilde{m}_-)\,(M_\odot/R_\odot)}{\tilde{R}^3} - \frac{3\,\tilde{M}^2\,(M_\odot/R_\odot)^2}{2\,\tilde{R}^4} - \left(\frac{6\,(\tilde{m}_+ - \tilde{m}_-)^2}{\tilde{M}^2} + \frac{\tilde{M}^2\,(M_\odot/R_\odot)^2}{2\,\tilde{R}^2}\right)\left(\frac{\tilde{M}'}{\tilde{M}}\right)^2 + \\ & \frac{\tilde{M}\,\tilde{M}'(M_\odot/R_\odot)^2}{\tilde{R}^3} + 2\left(\frac{(\tilde{m}_+ - \tilde{m}_-)^2}{\tilde{M}^2} - \frac{\tilde{M}^2\,(M_\odot/R_\odot)^2}{4\,\tilde{R}^2}\right)\left(\frac{\tilde{M}''}{\tilde{M}^2}\right),\end{aligned} \tag{26}$$

where $'$ stands for the derivative with respect to $\tilde{R}$. Computing the second derivative of M and substituting in this last expression that, as we mentioned above, is equivalent to evaluating the potential at the static solution (the results have been checked using MapleV-4, which has been used also in obtaining the figures), we obtain that

$$\frac{\partial^2 V}{\partial R^2}|_{R_{st}} = 0. \tag{27}$$

So, the perturbation equation for this case reduces to

$$(\ddot{\delta R}) = 0, \tag{28}$$

which in turn implies that the radial perturbations for the static shell are given by

$$\delta R = a\,\tau + b, \tag{29}$$

with a and b arbitrary constants, so they grow linearly and we thus conclude that the static shell is unstable under radial perturbations.

These results allow us to conclude that the potential for the static case has no minimum around the static radius but an inflection point so it has the form of a large plateau around that static radius and that under radial perturbations, which will grow linearly with respect to the proper time of the shell, will tend either to collapse or expand, depending on the sign of the initial conditions imposed for the perturbation. If $\delta\dot{R}(\tau_0) = a > 0$, it expands, and if $\delta\dot{R}(\tau_0) = a < 0$, it collapses.

4 Conclusions

In the present work we have presented the analysis of the equations of motion for a shell of perfect fluid using the formalism of General Relativity. We have chosen a model where the effects of the combined action of two type of forces acting on the shell are shown, namely the gravitational attraction versus a force due to the tangential stresses. Even though the problems has been reduced to a set of two coupled partial differential equations, an analytic solution was not found. Nevertheless the equations are very tractable and from the properties of the potential energy we have been able to present several types of motion allowed in our model, which depends on five parameters. We presented cases for oscillatory motion for masses of the order of a Solar mass and small radius, and for a radius of the order of Solar radius, which implies large masses. With an appropriate selection of the parameters for any value of mass or radius, such a motion can be found. We have also presented other two cases allowed by our model, one which could be interpreted as explosion, an ejection of a shell starting from rest due to the force associate with the tangential pressure, and another where that force equals the gravitational one, allowing the shell surrounding a cosmic body to be static. The matter in the shell is associated with an unusual equation of state, which does not violate any energy condition, so we can say that it is not an “exotic” type of matter but still unlikely to describe a realistic one, so it is actually a new, mathematically correct solution but the reader should be aware of the type of matter used in this model.

This last remark though, does not apply to the last case presented, the

static one, where we have obtained the model without any reference to an equation of state, so it holds for all of them. We also showed that the static shell is an unstable configuration under radial perturbations which grow linearly with respect to the proper time of the shell.

We can incorporate more parameters to describe other type of objects. For instance, we can take into account the radiation emitted from the inner body to the shell and the interaction of the shell with the medium, the radiation pressure and the friction type force respectively, considered in the classical formalism. This case could be obtained working with the matching conditions on the shell of a Vaidya universe inside (Vaidya P. C. 1951), and a Friedmann - Robertson - Walker dust universe outside (see Weinberg S. 1972). Also some deviations from sphericity could be considered, so that the matching conditions could be analyzed for an axisymmetric type of space times. This case is expected to be highly unstable, so the point would be to study the final state of that model. Finally, it would be of interest to study the motion of shells composed of different types of matter, such as one constructed from a scalar field. These ideas are currently under investigation.

We think that the model presented elucidates several features of the possible types of motion of the shell.

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5 Appendix

In this appendix we present a review of the description on the deduction of the equations of motion of the shell, Eqs.(2, 3). The four - dimensional space-time is taken to be composed of two parts $\mathcal{M}_-$ and $\mathcal{M}_+$, separated by a boundary Σ. The main goal in this analysis consists in showing how the geometry on Σ is determined by that of the space-time in which it is contained. Then, demanding continuity of the line elements on Σ, and analyzing the jump on the curvature due to the presence of the shell, to obtain the "matching" conditions or, as seeing in the present work, the equations of motion of the

shell.

The first condition for joining the two regions is that the line element of the region $\mathcal{M}_-$ should equal the line element of $\mathcal{M}_+$ at Σ, that is, both must be equal to $ds^2{}_\Sigma$.

The metric on the shell is described by γ_{ab}, (Latin indexes are 0,2,3, Greek 0,1,2,3). It is related to the metric of the space-time by $e^\alpha_{(a)}\, e^\beta_{(b)}\, \gamma^{ab} = g^{\alpha\beta} + n^\alpha\, n^\beta$, where n^α is a unit 4-dimensional vector normal to Σ, and the $e^\alpha_{(a)}$ are the projectors on the hypersurface: $e^\alpha_{(a)} A^a = A^\alpha$, $e^\alpha_{(a)} A_\alpha = A_a$, for an arbitrary vector $\mathbf{A}$. The 4-dimensional velocity is $u^\alpha \equiv \frac{dx^\alpha}{d\tau}$ and the 4-dimensional acceleration $\frac{\delta\, u^\alpha}{\delta \tau} = u^\alpha{}_{;\mu} u^\mu$, (semicolon denotes covariant derivative with respect to $g_{\alpha\beta}$).

One of the most important geometric quantities in the embedding is the extrinsic curvature defined by $K_{ab} = n_\beta\, e^\beta{}_{(a);\alpha}\, e^\alpha{}_{(b)}$. Recalling that $e^\beta{}_{(a)}\, n_\beta = 0$, it is not hard to see that the extrinsic curvature is related to the acceleration by

$$K_{ab} u^a\, u^b = n_\alpha \frac{\delta\, u^\alpha}{\delta \tau}. \tag{A.1}$$

So the extrinsic curvature projected on the 3-dimensional velocity equals to the projection of the 4-dimensional acceleration on the normal. The first matching condition refers to the difference betwwen the extrinsic curvature projected from $\mathcal{M}_+$ and the one obtained projecting from $\mathcal{M}_-$. Let us denote this difference by μ_{ab}:

$$\mu_{ab} = [K_{ab}] = K_{ab}|^+ - K_{ab}|^-. \tag{A.2}$$

where a quantity in square brackets denotes that difference of the projections: $[A] = A^+ - A^-$. From Eq. (A.2) together with Eq. (A.1), we can obtain an expression for the difference of the acceleration projected on the normal from the two regions of the space-time:

$$\left[n_\alpha \frac{\delta\, u^\alpha}{\delta \tau} \right] = \mu_{ab} u^a\, u^b, \tag{A.3}$$

Another important geometric quantity is the symmetric 3-tensor S_{ab}, defined by the "Lanczos equations"

$$\mu_{ab} - \gamma_{ab}\, \mu = 8\, \pi\, S_{ab},$$

or equivalently, $\mu_{ab} = 8\,\pi\,(S_{ab} - \frac{1}{2}\gamma_{ab}\,S)$, $(\mu = \gamma^{ab}\mu_{ab}, S = \gamma^{ab}S_{ab})$. The quantity S_{ab} can be considered to be the *surface energy tensor* of the shell, as it is the limit of the integral of the stress energy tensor through the thickness of the shell when this thickness tends to zero (Israel 1966). We will take the matter on the shell to be described by a perfect fluid,

$$S_{ab} = (\sigma + p)\,u_a\,u_b + p\gamma_{ab}, \qquad (A.4)$$

where σ is the matter-energy density on the shell, p is the tangential pressure of that matter, and u^a stands for the 3-dimensional time-like velocity vector, $\gamma_{ab}u^a\,u^b = -1$. Then, the equation Eq. (A.3) for the difference of the projections of the acceleration becomes

$$\left[n_\alpha\,\frac{\delta\,u^\alpha}{\delta\tau}\right] = \frac{M}{R^2} + 8\pi p, \qquad (A.5),$$

where M (not necessarily constant) is the gravitational mass of the shell, defined by $M = A\sigma$, with A the area of the shell, $A = 4\pi R^2$. Eq. (A.5) is the first of the matching equations, which tells us that a discontinuity of the normal component of the acceleration is related to the matter present on the shell.

There are two other matching equations which can be deduced starting from the relations of the components of the Riemann tensor in the space-time with the one on Σ. These relations are called the Codazzi-Mainardi equations:

$${}^4R^n{}_{bcd} = K_{bc|d} - K_{bd|c},$$

and the Gauss-Codazzi equations:

$${}^4R_{abcd} = {}^3R_{abcd} + K_{ac}\,K_{bd} - K_{ad}\,K_{bc},$$

where $|$ denotes covariant derivative on Σ, *i. e.*, $\gamma_{ab|c} = 0$. Multiplying the Codazzi-Mainardi equations by γ^{bc}, and the Gauss-Codazzi by $\gamma^{ac}\,\gamma^{bd}$, we get a set of four equations, valid on each side of the shell:

$$G_{\mu\nu}\,n^\mu\,e^\nu{}_{(a)}\,|^\pm = (K^b{}_{a|b} - K_{,a}\,)|^\pm,$$

$$2G_{\mu\nu}\,n^\mu\,n^\nu\,|^\pm = {}^3R + (K^2 - K_{ab}\,K^{ab})|^\pm,$$

With these definitions and the Einstein equations, $G_{\mu\nu} = 8\,\pi\,T_{\mu\nu}$, the difference of the projections of the stress-energy tensor evaluated on the shell are:

$$[T_{\mu\nu}\,n^{\mu}\,u^{\nu}] = u^{(a)}\,S^{b}{}_{a|b}, \tag{A.6}$$

$$2\,[T_{\mu\nu}\,n^{\mu}\,n^{\nu}] = -S^{ab}(K^{+}_{ab} + K^{-}_{ab}), \tag{A.7}$$

which are the other two matching equations, relating the projections of the stress energy tensor on the normal and on the velocity and on the normal on both indexes to the matter distribution on the shell. Eq. (A.6) is the equation of energy balance that tells us how the fluxes of matter-energy from both sides determine the dynamics of the shell. On the other hand, Eq. (A.7) has no clear physical interpretation (see below).

For the matter-energy of the shell described by a perfect fluid, and taking into account the fact that $\dot{\sigma} \equiv \frac{d\sigma}{d\tau}$ and that $u^{b}{}_{|b} = \frac{1}{\sqrt{-\gamma}}((\sqrt{-\gamma}\,u^{b})_{,b}$, with $\gamma \equiv det(\gamma_{ab})$, as well as that for spherical symmetry, the line element of the shell is given by Eq. (1). The shell Σ is a 3 dimensional manifold, characterized by a line element

$$ds^{2}{}_{\Sigma} = -d\tau^{2} + R^{2}(\tau)d\Omega^{2}, \tag{A.8}$$

with $R(\tau)$ the radius of the shell. Thus we can obtain that $u^{b}{}_{|b} = \frac{\dot{A}}{A}$, with A the area of the shell defined above, the energy balance equation, Eq. (A.6), takes the form

$$\dot{M} + p\dot{A} = 4\,\pi\,R^{2}\;[T_{\mu\nu}\,n^{\mu}\,u^{\nu}]\,, \tag{A.9}$$

which is the form of the second matching condition (Núñez and Oliveira 1996). The interpretation as the equation of motion for the shell looks particularly clear in this form. Here it is important to stress the fact that with spherical symmetry and a perfect fluid, the first matching condition (A.5) can be integrated without further assumptions, using Eq. (A.9), to yield a quadratic first order equation for $R(\tau)$, Eq. (3). For details on this deduction see (Núñez and Oliveira 1996).

Finally, the third matching condition (A.7) which, using Eq. (A.1) and S^{ab} as a perfect fluid, Eq. (A.4), tells us that the sum of the projections of the acceleration on the normal, from each side of Σ is given by

$$\sigma\,\{n_{\alpha}\,\frac{\delta\,u^{\alpha}}{\delta\tau}\} \equiv \sigma\,(n_{\alpha}\,\frac{\delta\,u^{\alpha}}{\delta\tau}|^{+} + n_{\alpha}\,\frac{\delta\,u^{\alpha}}{\delta\tau}|^{-}) = -2\,[T_{\mu\nu}\,n^{\mu}\,n^{\nu}] - p\,\{K_{ab}u^{a}u^{b} + K\,\}. \tag{A.10}$$

As mentioned above, this equation has no clear physical meaning. As a matter of fact, it can be shown that, at least for the spherical case with the matter-energy of the shell described by a perfect fluid, Eq. (A.4), this last equation Eq. (A.11) is the same as the matching equation Eq. (A.5), so we do not have to consider it in the analysis of the motion of the shell. For details on the demonstration of this equality see (Núñez, 1996).

In this way, the motion equations for the shell reduce to a set of two first order equations, namely Eqs. (2, 3). The equation for the change in the advanced time can be obtained from the equation for the change in the radius, using the normalization equation for the cuadrivelocity. The problem is completed with a equation of state for the matter-energy of the shell, $p = p(\sigma)$.

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